

THE DRIFT LAPLACIAN AND HERMITIAN GEOMETRY

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ABSTRACT. Let (M^n, h) be a compact Hermitian manifold. Suppose λ is the lowest eigenvalue of the complex Laplacian on M . We prove that $\lambda \geq C$ where C depends only on the dimension n , the diameter d , the Ricci curvature of the Levi-Civita connection on M , and a norm, expressed in curvature, that determines how much M fails to be Kähler. We first estimate the principal eigenvalue of a drift Laplacian and then study the structure of Hermitian manifolds using recent results due to Yang and Zheng [18]. We combine these results to obtain the main estimate.

1. INTRODUCTION

This preprint's main goal is to obtain a lower bound on the spectrum of the complex Laplacian on a compact Hermitian manifold. To do this, we need two seemingly unrelated results. First, we derive an estimate for the principal eigenvalue of a Laplacian with drift. Second, using recent results from [18], we find inequalities which allow us to estimate the torsion of a Hermitian manifold in terms of the Riemannian and Hermitian curvature. We then note that the complex Laplacian can be viewed as a drift Laplacian in which the drift can be expressed in terms of the torsion. We thus get the desired estimate.

In Section 1, we state and discuss the results. In Section 2, we prove the estimate for the drift Laplacian. In Section 3, we discuss the torsion in greater depth and prove the lemmas in Section 1 that we need to bound the torsion. In the final section, we prove the estimate on the complex Laplacian and discuss some conjectures on the relation between curvature tensors, the torsion tensor, and orthogonal complex structures.

1.1. The Drift Laplacian. The drift Laplacian is a natural operator that appears in physical applications. The associated heat equation has been studied and the drift term acts as convection (i.e. stirring). Often, though not always, stirring speeds up the diffusion process. Therefore, we might expect to be able to derive lower bounds on the spectrum for the drift Laplacian. Theorem 1 provides an example of such bounds.

Theorem 1. *Let (M^n, g) be a compact Riemannian manifold without boundary, satisfying $\text{Ric } M \geq -(n-1)k$ and ξ be a one-form on M . Suppose $u \in C^\infty(M)$ is a solution to the equation $\Delta u + \xi(\nabla u) = \lambda u$. Let $|\xi|$ be the C_0 norm of ξ and $|\nabla \xi|$ be the C_0 norm of $\nabla \xi$ (as a two tensor). Let $D = 2nd^2$ where d is the diameter of M and $E = \frac{1}{2n}((16n^2 - 32n + 5)|\xi|^2 + 2(n-1)^2k + 2(n-1)|\nabla \xi|)$. Then we have:*

$$\lambda \geq \frac{1}{D} \frac{(1 + \sqrt{1 + 4DE})^2 - DE}{\exp(1 + \sqrt{1 + 4DE})}$$

Similar (and sharper) estimates have been obtained in the case of the Witten-Laplacian, where $\xi = df$ for some smooth function f , such as in [1] [6]. For our purposes, ξ will generally not be exact so we cannot use these results. The one-form is exact if and only if the metric is conformal to a balanced metric, which is a very restrictive condition. We would not be surprised if Theorem 1 were already known but we have not been able to find it in our literature search. In [9], Gonzalez and Negrin study the kernel of the drift Laplacian on open domains with the same conditions that we use. More recently, Jorgen Jost and others have studied harmonic maps for a generalization of this operator (so called V-Harmonic maps). This group has proven various results and advanced the theory of harmonic maps [4].

1.2. Structural inequalities on Hermitian manifolds. In Section 3, we use recent results from [18] to derive inequalities that estimate the torsion of a Hermitian manifold in terms of the Riemannian and Hermitian curvature. We define two norms that measure the difference between Hermitian and Riemannian curvature, denoted R^h and R , respectively.

Given a unitary frame $\{e_i\}$ on a Hermitian manifold, we define $|R^h - R|_\star^2$ and $|R^h - R|_{\star\star}^2$ in the following way:

$$\begin{aligned} |R^h - R|_\star^2 &= \sum_{i,j,k,l} |R_{i\bar{j}k\bar{l}}^h - R_{i\bar{j}k\bar{l}}|^2 + 2 \sum_{i,j,k,l} |R_{i\bar{j}k\bar{l}}|^2 \\ |R^h - R|_{\star\star}^2 &= \sum_{i,j,k,l} |R_{i\bar{j}k\bar{l}}|^2 \end{aligned}$$

We show that these quantities dominate the C^0 and C^1 norm of the torsion. To be precise, let $\nabla^{c'}$ and $\nabla^{c''}$ be the $(1,0)$ and $(0,1)$ components of the covariant differentiation of the Chern connection defined by:

$$\begin{aligned} \nabla_{X+\bar{Y}}^{c'} T &= \nabla_X^c T \text{ and} \\ \nabla_{X+\bar{Y}}^{c''} T &= \nabla_{\bar{Y}}^c T \end{aligned}$$

where X and Y are any complex tangent vectors on M of type $(1,0)$.

Theorem 2. *The following inequalities hold pointwise:*

$$\begin{aligned} \|T\|^2 &\leq |R^h - R|_\star \text{ and } \|\eta\|^2 \leq |R^h - R|_\star \\ |\nabla^{c'}(T)| &\leq |R^h - R|_\star \\ |\nabla^{c''}T| &\leq C(n)|R^h - R|_\star + |R^h - R|_{\star\star} \end{aligned}$$

Here, η is Gauduchon's torsion one-form, defined by $\partial\omega^{n-1} = -2\eta \wedge \omega^{n-1}$, where ω is the Kähler (metric) form of the metric h . Given a unitary frame $\{e_i\}$, we can also define η as $\eta_i = \sum_j T_{ij}^j$.

The torsion expresses the difference between the Levi-Civita connection and the Hermitian connection. Therefore, given a unit vector X of type $(1,0)$, the difference between $\nabla_X^{c''}T$ and $\nabla_{\bar{X}}T$ can be bounded by a quadratic expression in torsion. We can bound the difference between $\nabla_X^{c'}T$ and $\nabla_X T$ in the same way. Using these observations, we obtain Theorem 3.

Theorem 3. *Let T be the torsion tensor and ∇T the derivative of the torsion tensor with respect to the Levi-Civita connection. Then there exists $C'(n)$ so that following inequality holds:*

$$\|\nabla T\| \leq C'(n)|R^h - R|_\star + |R^h - R|_{\star\star}$$

1.3. The Complex Laplacian. Using the observation that the complex Laplacian on a Hermitian manifold can be expressed as a Laplacian with drift, we translate our estimate on the eigenvalue on the drift Laplacian into an estimate on the Laplacian on a Hermitian manifold.

Theorem 4. *Suppose that (M^n, h) is a compact, Hermitian manifold. Then there exists a uniform $C > 0$ such that:*

$$\lambda \geq \frac{1}{4n} \frac{\left(\frac{2}{d^2} + 3Cn^2(k + |R - R^h|_\star + |R - R^h|_{\star\star})\right)}{\exp\left(1 + \sqrt{1 + 4Cn^2d^2(k + |R - R^h|_\star + |R - R^h|_{\star\star})}\right)}$$

This estimate is unsightly, but only involves the dimension, the diameter, the Ricci curvature, and the norms we defined earlier. Furthermore, the estimate scales as expected. Spectral geometry of Hermitian manifolds has been studied [14] [8], especially in the context of finding spectral conditions which ensure a Hermitian manifold is balanced or Kähler. Our results suggest that one can understand the spectral geometry of Hermitian manifolds by studying the torsion. In a future preprint, we will try to strengthen these estimates and prove other results in this vein. We put forth the following conjecture that this estimate can be improved to only involve the Riemannian curvature tensor.

Conjecture 5. *Given a compact Hermitian manifold (M^n, h) , there exists C depending only on the dimension, diameter, volume, and Riemannian curvature tensor such that if $\square u = \lambda u$ then $\lambda \geq C$.*

This would mirror the case for the Laplace-Beltrami operator, where an estimate exists in terms of the dimension, diameter, and Ricci curvature. This conjecture can be reduced to the following estimates on the torsion one-form.

Conjecture 6. *Given a compact Hermitian manifold (M^n, h) , there exist constants C and C'' depending only on the dimension, diameter, volume, and Riemannian curvature tensor such that if $|\eta| \leq C$ and $|\nabla^{c''} \eta| \leq C''$.*

We can show that there are certain curvature conditions which force η to vanish, but we have not been able to establish this for η non-zero. However, even if the latter estimate fails, the first conjecture may still be true if a Faber-Krahn type inequality can be established. The Laplacian with drift has been studied on smoothly-bounded domains in \mathbb{R}^n . In this setting, the following paper proves a version of the Faber-Krahn inequality that only relies on the C_0 norm of the drift [10]. If some version of this result were true on compact manifolds, one could use this result to study Conjecture 5. However, Faber-Krahn type theorems usually rely on the isoperimetric inequality, which provides an obstacle to direct application in the context of compact manifolds.

For future work, we hope to continue studying torsion and to try to understand the moduli space of complex structures which are orthogonal to a given Riemannian metric. This would show how much information the Riemannian geometry can detect about the complex structure. To give another result in this vein, Gauduchon proved that hyperbolic manifolds of dimension great than two do not admit complex structures, [8], a result that Hernandez-lamonedá extended to hold for negative strictly quarter-pinched manifolds [11]. However, one would hope interesting results also hold when the moduli space is not just the empty set.

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2. ESTIMATING THE PRINCIPAL EIGENVECTOR OF THE DRIFT LAPLACIAN

We now prove Theorem 1, which gives an estimate for the principal eigenvalue of the drift Laplacian $L = \Delta + \xi(\nabla)$. This proof is an adaptation of the one given in Schoen and Yau's Lectures in Differential Geometry [16] for the principal eigenvalue of the Laplacian. Recall that Theorem 1 states the following:

Theorem. *Suppose (M^n, g) is a compact Riemannian manifold without boundary satisfying $\text{Ric } M \geq -(n-1)k$ for $k \geq 0$ and diameter d . Suppose that u satisfies:*

$$Lu + \lambda u = 0$$

Let $|\xi|$ be the C_0 norm of ξ and $|\nabla\xi|$ be the C_0 norm of $\nabla\xi$ (as a two-tensor). If $D = 2nd^2$ where d is the diameter of M and $E = \frac{1}{2n}((16n^2 - 32n + 5)|\xi|^2 + 2(n-1)^2k + 2(n-1)|\nabla\xi|)$. Then we have

$$\lambda \geq \frac{1}{D} \frac{(1 + \sqrt{1 + 4DE})^2 - DE}{\exp(1 + \sqrt{1 + 4DE})}$$

Proof. The proof uses the standard technique of utilizing a Bochner identity on a family of functions $G_\beta(x)$ to establish a gradient estimate and then integrating the estimate to obtain an inequality involving λ (as well as the other terms that appear). Finally, we solve the inequality in terms of λ and optimize the estimate in terms of β to get the desired result. However, the details in the calculation are somewhat messy.

Suppose u satisfies:

$$(1) \quad \Delta u + \xi(\nabla u) + \lambda u = 0$$

with $\sup u = 1$. Let $\beta > 1$ and consider the function $G(x)$ defined by:

$$(2) \quad G(x) = \frac{|\nabla u|^2}{(\beta - u)^2}$$

We will establish an estimate on $G(x)$ and use this to derive an estimate on u .

2.1. Set up using the Bochner formula. Then suppose that $G(x)$ is maximized at x_0 . We have that $\nabla G(x_0) = 0$ and $\Delta G(x_0) \leq 0$.

Also, we have $G(x)(\beta - u)^2 = |\nabla u|^2$ so:

$$(\Delta G)(\beta - u)^2 + 2\nabla G \nabla(\beta - u)^2 + G\Delta(\beta - u)^2 = \Delta|\nabla u|^2$$

At x_0 , we have $\nabla G(x_0) = 0$ so using the Bochner formula in normal coordinates at x_0 , we have the following:

$$\begin{aligned}
0 &\geq \Delta|\nabla u|^2 - G\Delta(\beta - u)^2 \\
&= 2 \sum_{i,j} u_{ij}^2 + 2 \sum_i u_i(\Delta u)_i + 2Ric(\nabla u, \nabla u) - 2G((-\Delta u)(\beta - u) + |\nabla u|^2)
\end{aligned}$$

Then dividing by 2, using (1) and the curvature bounds, we have that:

$$\begin{aligned}
0 &\geq \sum_{i,j} u_{ij}^2 + \sum_i u_i(-\xi(\nabla u) - \lambda u)_i + Ric(\nabla u, \nabla u) - G((-\Delta u)(\beta - u) + |\nabla u|^2) \\
&\geq \sum_{i,j} u_{ij}^2 + \sum_i u_i(-\xi(\nabla u) - \lambda u)_i - (n-1)k|\nabla u|^2 - G((-\Delta u)(\beta - u) + |\nabla u|^2)
\end{aligned}$$

We pick normal coordinates at x_0 so that $u_i = 0$ for $i > 1$ and $u_1 = |\nabla u|$. Then $\nabla G(x_0) = 0$ implies:

$$(3) \quad u_{11} = \frac{-|\nabla u|^2}{\beta - u} \text{ and } u_{1i} = 0 \text{ otherwise.}$$

Let $\mathcal{G} = G((\xi(\nabla u) + \lambda u)(\beta - u) + |\nabla u|^2)$. In the following manipulation we will not change this term so we use \mathcal{G} as shorthand. Then, at x_0 :

$$\begin{aligned}
0 &\geq \sum_{i,j} u_{ij}^2 + \sum_i u_i(-\xi(\nabla u) - \lambda u)_i - (n-1)k|\nabla u|^2 - G((\xi(\nabla u) + \lambda u)(\beta - u) + |\nabla u|^2) \\
&= \sum_{i,j} u_{ij}^2 + u_1(-\xi(\nabla u) - \lambda u)_1 - (n-1)k|\nabla u|^2 - \mathcal{G} \\
&= \sum_{i,j} u_{ij}^2 - u_1(\xi(\nabla u))_1 - \lambda u_1^2 - (n-1)k|\nabla u|^2 - \mathcal{G} \\
&= \sum_{i,j} u_{ij}^2 - u_1\xi(e_1)u_{11} - \xi(e_1)_1u_1^2 - \lambda u_1^2 - (n-1)k|\nabla u|^2 - \mathcal{G} \\
&\geq \sum_{i,j} u_{ij}^2 - u_1|\xi||u_{11}| - |\nabla \xi|u_1^2 - \lambda u_1^2 - (n-1)k|\nabla u|^2 - \mathcal{G} \\
&= \sum_{i,j} u_{ij}^2 - u_1|\xi||u_{11}| - (|\nabla \xi| + \lambda + (n-1)k)|\nabla u|^2 - \mathcal{G}
\end{aligned}$$

Therefore, we have:

$$(4) \quad 0 \geq \sum_{i,j} u_{ij}^2 - u_1|\xi||u_{11}| - (|\nabla \xi| + \lambda + (n-1)k)|\nabla u|^2 - \mathcal{G}$$

2.2. Putting the second derivatives to use. Continuing to work at x_0 , we now note that:

$$\begin{aligned}
\sum_{i,j=2}^n u_{ij}^2 &\geq \sum_{i=2}^n u_{ii}^2 \\
&\geq \frac{1}{n-1} \left(\sum_{i=2}^n u_{ii} \right)^2 \\
&= \frac{1}{n-1} (\Delta u - u_{11})^2 \\
&= \frac{1}{n-1} (-\xi(\nabla u) - \lambda u - u_{11})^2 \\
&= \frac{1}{n-1} (\xi(u_1) + \lambda u + u_{11})^2 \\
&\geq \frac{1}{n-1} \left(\frac{u_{11}^2}{2} - (\xi(u_1) + \lambda u)^2 \right) \\
&\geq \frac{1}{n-1} \left(\frac{u_{11}^2}{2} - 2(\xi(u_1))^2 - 2(\lambda u)^2 \right)
\end{aligned}$$

Substituting this inequality into (4), we get the following:

$$\begin{aligned}
0 \geq u_{11}^2 + \frac{1}{n-1} \left(\frac{u_{11}^2}{2} - 2(\xi(u_1))^2 - 2(\lambda u)^2 \right) \\
- u_1 |\xi| |u_{11}| - (|\nabla \xi| + \lambda + (n-1)k) |\nabla u|^2 - \mathcal{G}
\end{aligned}$$

Using (2) and the definition of \mathcal{G} , we have:

$$\begin{aligned}
0 \geq u_{11}^2 + \frac{1}{n-1} \left(\frac{u_{11}^2}{2} - 2(\xi(u_1))^2 - 2(\lambda u)^2 \right) - u_1 |\xi| |u_{11}| \\
- (|\nabla \xi| + \lambda + (n-1)k) |\nabla u|^2 - \frac{|\nabla u|^2}{(\beta - u)^2} ((\xi(\nabla u) + \lambda u)(\beta - u) + |\nabla u|^2)
\end{aligned}$$

Then by (3), the first and last terms cancel, leaving:

$$\begin{aligned}
0 &\geq \frac{1}{n-1} \left(\frac{u_{11}^2}{2} - 2(\xi(u_1))^2 - 2(\lambda u)^2 \right) - u_1 |\xi| |u_{11}| \\
&\quad - (|\nabla \xi| + \lambda + (n-1)k) |\nabla u|^2 - \frac{|\nabla u|^2}{(\beta-u)^2} (\xi(\nabla u) + \lambda u) (\beta-u) \\
&\geq \frac{1}{2(n-1)} u_{11}^2 - u_1 |\xi| |u_{11}| - \left(|\nabla \xi| + \lambda + (n-1)k + \frac{2}{n-1} |\xi|^2 \right) |\nabla u|^2 \\
&\quad - \frac{|\nabla u|^2}{(\beta-u)} (\xi(\nabla u) + \lambda u) - \frac{2}{n-1} \lambda^2 u^2 \\
&\geq \frac{1}{2(n-1)} u_{11}^2 - u_1 |\xi| |u_{11}| - \left(|\nabla \xi| + \lambda + (n-1)k + \frac{2}{n-1} |\xi|^2 \right) |\nabla u|^2 \\
&\quad - \frac{|\nabla u|^2}{(\beta-u)} |\xi| |\nabla u| - \lambda u \frac{|\nabla u|^2}{(\beta-u)} - \frac{2}{n-1} \lambda^2 u^2 \\
&= \frac{1}{2(n-1)} \frac{|\nabla u|^4}{(\beta-u)^2} - 2|\xi| \frac{|\nabla u|^3}{(\beta-u)} - \left(|\nabla \xi| + \lambda + (n-1)k + \frac{2}{n-1} |\xi|^2 \right) |\nabla u|^2 \\
&\quad - \lambda u \frac{|\nabla u|^2}{(\beta-u)} - \frac{2}{n-1} \lambda^2 u^2
\end{aligned}$$

Now we divide this inequality by $(\beta-u)^2$ to obtain:

$$\begin{aligned}
0 &\geq \frac{1}{2(n-1)} \frac{|\nabla u|^4}{(\beta-u)^4} - 2|\xi| \frac{|\nabla u|^3}{(\beta-u)^3} - \left(|\nabla \xi| + \lambda + (n-1)k + \frac{2}{n-1} |\xi|^2 \right) \frac{|\nabla u|^2}{(\beta-u)^2} \\
&\quad - \lambda u \frac{|\nabla u|^2}{(\beta-u)^3} - \frac{2}{n-1} \lambda^2 \frac{u^2}{(\beta-u)^2}
\end{aligned}$$

Let $\alpha = \frac{u}{\beta-u}$ and note that $\alpha \leq \frac{1}{\beta-u} \leq \frac{1}{\beta-1}$.

Then we can rewrite this inequality in terms of G and α as:

$$\begin{aligned}
(5) \quad 0 &\geq \frac{1}{2(n-1)} G(x_0)^2 - 2|\xi| G^{3/2} - \left(|\nabla \xi| + \lambda + (n-1)k + \frac{2}{n-1} |\xi|^2 \right) G(x_0) - \lambda \alpha G(x_0) - \frac{2}{n-1} \lambda^2 \alpha^2
\end{aligned}$$

Since x_0 maximizes $G(x_0)$, this inequality holds true for G throughout M .

2.3. Deriving and avoiding a quartic equation. By (5), we have

$$\begin{aligned}
0 &\geq G^2 - 4(n-1)|\xi|G^{3/2} - 2(n-1)\left(|\nabla\xi| + \lambda + (n-1)k + \frac{2}{n-1}|\xi|^2 + \lambda\alpha\right)G - 4\lambda^2\alpha^2 \\
&= G^2 - 4(n-1)|\xi|G^{3/2} - 2(n-1)\left(|\nabla\xi| + (n-1)k + \frac{2}{n-1}|\xi|^2 + \lambda(\alpha+1)\right)G - 4\lambda^2\alpha^2 \\
&\geq G^2 - 4(n-1)|\xi|G^{3/2} - 2(n-1)\left(|\nabla\xi| + (n-1)k + \frac{2}{n-1}|\xi|^2 + \frac{\beta}{\beta-1}\lambda\right)G - 4\lambda^2\alpha^2
\end{aligned}$$

Letting $g = \sqrt{G}$, and $A = 4(n-1)|\xi|$, $B = 2(n-1)\left(|\nabla\xi| + \frac{\beta}{\beta-1}\lambda + (n-1)k + \frac{2}{n-1}|\xi|^2\right)$ and $C = 4\lambda^2\alpha^2$, this reduces to:

$$(6) \quad 0 \geq g^4 - Ag^3 - Bg^2 - C$$

We could try to solve this quartic and then use that to get estimates on λ . However, it is much more straightforward to try to estimate g .

Lemma 7. *Given $A, B, C > 0$, if x satisfies $P(x) = x^4 - Ax^3 - Bx^2 - C \leq 0$, Then $x \leq A + \sqrt{B + \sqrt{C}} = a$.*

Note that $P(x) + C = x^2(x^2 - Ax - B)$ so it is sufficient to show that $P(a) > 0$, because increasing x will only increase both factors if the latter is already positive.

Then we have the following:

$$\begin{aligned}
P(a) &= (A + \sqrt{B + \sqrt{C}})^4 - A(A + \sqrt{B + \sqrt{C}})^3 - B(A + \sqrt{B + \sqrt{C}})^2 - C \\
&= (A + \sqrt{B + \sqrt{C}})^2 \left((A + \sqrt{B + \sqrt{C}})^2 - A(A + \sqrt{B + \sqrt{C}}) - B \right) - C \\
&= (A + \sqrt{B + \sqrt{C}})^2 \left(A^2 + B + \sqrt{C} + 2A\sqrt{B + \sqrt{C}} - A^2 - A\sqrt{B + \sqrt{C}} - B \right) - C \\
&= (A + \sqrt{B + \sqrt{C}})^2 \left(\sqrt{C} + A\sqrt{B + \sqrt{C}} \right) - C > 0
\end{aligned}$$

Thus the lemma is proved. However, in order to make future calculations more feasible, we note the following inequality holds:

$$A + \sqrt{B + \sqrt{C}} \leq 2\sqrt{A^2 + B + \sqrt{C}}$$

Using $2\sqrt{A^2 + B + \sqrt{C}}$ for g in our problem now, we obtain:

$$\begin{aligned}
g &\leq \sqrt{16(n-1)^2|\xi|^2 + 2(n-1)\left(|\nabla\xi| + \frac{\beta}{\beta-1}\lambda + (n-1)k + \frac{2}{n-1}|\xi|^2\right) + \sqrt{4\lambda^2\alpha^2}} \\
&= \sqrt{(16(n-1)^2 + 4)|\xi|^2 + 2(n-1)^2k + 2(n-1)|\nabla\xi| + (2(n-1)\frac{\beta}{\beta-1} + 2\alpha)\lambda} \\
&\leq \sqrt{(16n^2 - 32n + 5)|\xi|^2 + 2(n-1)^2k + 2(n-1)|\nabla\xi| + 2(\frac{\beta(n-1) + 1}{\beta-1})\lambda} \\
&\leq \sqrt{(16n^2 - 32n + 5)|\xi|^2 + 2(n-1)^2k + 2(n-1)|\nabla\xi| + 2n(\frac{\beta}{\beta-1})\lambda}
\end{aligned}$$

Recalling the definition of g , we can write this inequality as:

$$(7) \quad |\nabla u| \leq (\beta - u) \sqrt{(16n^2 - 32n + 5)|\xi|^2 + 2(n-1)^2k + 2(n-1)|\nabla\xi| + 2n(\frac{\beta}{\beta-1})\lambda}$$

2.4. Getting an inequality on λ . Now that we have a gradient estimate, we are most of the way done. It remains to integrate the inequality to get a C^0 estimate and pick β so that this gives us a useful inequality on λ .

Take $x_1, x_2 \in M$ such that $u(x_1) = 0$ and $u(x_2) = 1$. Let γ be the shortest geodesic joining x_1 and x_2 . Let d be the diameter of M . Note that the geodesics and diameter are defined in terms of the Levi-Civita connection because the length of paths depends only on the metric. We will discuss this phenomena further in future preprints.

Then:

$$\begin{aligned}
\log \frac{\beta}{\beta-1} &\leq \int_{\gamma} \frac{|\nabla u|}{\beta-u} \leq \\
&\leq d \sqrt{(16n^2 - 32n + 5)|\xi|^2 + 2(n-1)^2k + 2(n-1)|\nabla\xi| + 2n(\frac{\beta}{\beta-1})\lambda}
\end{aligned}$$

That is to say:

$$\lambda \geq \frac{\beta-1}{2n\beta} \left(\frac{1}{d^2} (\log \frac{\beta}{\beta-1})^2 - (16n^2 - 32n + 5)|\xi|^2 - 2(n-1)^2k - 2(n-1)|\nabla\xi| \right)$$

Let

$$E = \frac{1}{2n} ((16n^2 - 32n + 5)|\xi|^2 + 2(n-1)^2k + 2(n-1)|\nabla\xi|),$$

$$D = 2nd^2,$$

$$\text{and } x = \frac{\beta}{\beta - 1}$$

Then this boils down to:

$$\lambda \geq \frac{1}{D} \frac{(\log x)^2}{x} - \frac{E}{x} = f(x)$$

2.5. Strengthening the inequality. Taking the derivative of this with respect to x , we find:

$$f'(x) = \frac{1}{Dx^2}(\log x - (\log x)^2 + DE)$$

This is zero if $\log(x) = 1 + \sqrt{1 + 4DE}$, which is the value of x which maximizes the right hand side. Finally, this implies that:

$$\lambda \geq \frac{1}{D} \frac{(1 + \sqrt{1 + 4DE})^2 - DE}{\exp(1 + \sqrt{1 + 4DE})}$$

□

Despite how complicated the estimate is, notice that the quantity scales correctly under the scalar deformation ρg where ρ is a positive constant. This estimate is not optimal. There are a few places that this can be improved. We did not solve the quartic equation exactly (and if we had, inverting to solve for λ would be unpleasant). Furthermore, the integration essentially assumes that $G(x)$ is constant. It does not effectively use a barrier function as in the theorem due to Zhong-Yang which derives optimal eigenvalue bounds for compact Riemannian manifolds with $Ric(M) \geq 0$ [16]. We imagine that such an estimate can be improved using the various methods of [3], which we may attempt to do in the future. Also, the effect of the drift is overstated. There is no way that $(\xi(e_1))_1 = |\nabla \xi|$ and $|\xi| = \xi(e_1)$ and both quantities are maximized along the entire curve that we integrate along.

3. THE COMPLEX LAPLACIAN ON A HERMITIAN MANIFOLD

We now apply the previous estimate to complex geometry. The complex Laplacian on a Hermitian manifold can naturally be written as a Laplacian with drift equation. If η is the torsion one-form as before (η_i is the i -th component of the form as opposed to the derivative) and ξ is the Lee form, then we have the following:

$$(8) \quad \square f = \frac{1}{2} \Delta f + \xi(\nabla f)$$

$$(9) \quad \text{and } \eta + \bar{\eta} = -2\xi$$

We believe that it is well known at this point, but it is worth noting that $\Delta = 2\Box$ if $\eta = 0$, which is to say that (M, h) is a balanced metric. Therefore, a sufficient condition for the Laplacian and twice the complex Laplacian to be isospectral is that the metric is balanced. Another condition ensuring that the metric is balanced is that $\xi = 0$, and ξ is exact if and only if the metric is conformal to a balanced metric. Therefore, the estimates from the Witten-Laplacian can only be used in the special case where the metric is conformally equivalent to a balanced metric.

Conjecture 8. *Given, (M^n, h) a complex manifold, if Δ and $2\Box$ have the same spectrum, then (M^n, h) is balanced.*

Any counterexample would be a very interesting Hermitian manifold in its own right. It is known that if the complex Laplacian on a manifold is isospectral to the complex Laplacian on a balanced manifold, then it must be balanced as well. H. Donnelly showed in [5] that if \Box and Δ are isospectral on functions and one-forms, that the manifold is in fact Kähler. This is very similar to Peter Gilkey's result that if two complex manifolds are isospectral, they are either both Kähler or neither is [8].

However, we turn our attention away from the cases in which η is zero and try to understand it for general Hermitian manifolds. From the Theorem 1, if we have control over ξ and $\nabla\xi$ (where ∇ is with respect to the Levi-Civita connection), then we will be able to obtain bounds on the spectrum of \Box . By equation 9, this reduces to finding estimates on η and $\nabla\eta$. We now derive such estimates in terms of the curvature of the Levi-Civita and Chern connections.

3.1. Structural Inequalities on Hermitian Manifolds. From a geometric (and non-rigorous) point of view, the only metric invariants should come from curvature so the need for twisting of a unitary frame can be entirely determined by how “non-flat” the underlying space is. Furthermore, the deformation of shapes and angles occurs due to the presence of curvature, not its derivatives, so we expect the torsion is bounded somehow by the curvature and how the curvature of the Chern and Levi-Civita connections differ. This phenomena has been noted before and studied in a somewhat different context in [15], which shows that if a complex structure exists at all, certain parts of the Weyl tensor must vanish. These observations suggest that torsion should be at most a “zero-th” order phenomena with respect to the curvature and we seek to formalize this intuition. The following relies heavily on the results from [18], which we cite repeatedly.

By equation 41 of [18], we have that given any type $(1, 0)$ vector X ,

$$R_{X\bar{X}X\bar{X}}^h - R_{X\bar{X}X\bar{X}} = \sum_k |T_{kX}^X|^2$$

Therefore, in any unitary frame $\{e_i\}$,

$$\begin{aligned} \sum_{i=1}^n R_{i\bar{i}i\bar{i}}^h - R_{i\bar{i}i\bar{i}} &= \sum_{i=1}^n \sum_k |T_{ki}^i|^2 \\ &= \sum_{k=1}^n \sum_i |T_{ki}^i|^2 \\ &\geq \frac{1}{(n-1)} \sum_{k=1}^n |\eta_k|^2 \\ &= \frac{1}{(n-1)} \|\eta\|_2^2 \end{aligned}$$

However, we can get stronger estimates using the structure theorems of [18].

For clarity, we write the results of Lemma 7 of [18] here.

Theorem. (Lemma 7) *Let (M^n, g) be a Hermitian manifold and let $p \in M$. Let $\{e_i\}$ be a unitary frame near p such that $\theta|_p = 0$. Then, at the point p we have:*

$$\begin{aligned} 2T_{ij,\bar{l}}^k &= R_{j\bar{l}i\bar{k}}^h - R_{i\bar{l}j\bar{k}}^h \\ T_{ij,k}^l &= R_{ijk\bar{l}}^r - T_{ri}^l T_{jk}^r + T_{rj}^l T_{ik}^r \\ 2R_{ij\bar{k}\bar{l}} &= T_{ij,\bar{k}}^l - T_{ij,\bar{l}}^k + 2T_{ij}^r \overline{T_{kl}^r} + T_{ri}^k \overline{T_{rl}^j} + T_{rj}^l \overline{T_{rk}^i} - T_{ri}^l \overline{T_{rk}^j} - T_{rj}^k \overline{T_{rl}^i} \\ R_{k\bar{l}i\bar{j}} &= R_{k\bar{l}i\bar{j}}^h - T_{ik,\bar{l}}^j - \overline{T_{jl,\bar{k}}^i} + T_{ik}^r \overline{T_{jl}^r} - T_{rk}^j \overline{T_{rl}^i} - T_{ri}^l \overline{T_{rj}^k} \end{aligned}$$

where r is summed through and $h_{,i} = e_i(h)$ and $h_{,\bar{i}} = \bar{e}_i(h)$.

Now we recall the two norms that measure how much the metric fails to be Kähler. We define $|R^h - R|_\star^2$ as:

$$|R^h - R|_\star^2 = \sum_{i,j,k,l} |R_{ijk\bar{l}}^h - R_{i\bar{j}k\bar{l}}|^2 + 2 \sum_{i,j,k,l} |R_{ij\bar{k}\bar{l}}|^2$$

Recall that $R_{XY\bar{Z}\bar{W}}^h = 0$ by Gray's theorem so this can be thought of as a norm of the differences of Riemannian and Hermitian curvatures.

Also, we define $|R^h - R|_{\star\star}^2$ in the following way:

$$|R^h - R|_{\star\star}^2 = \sum_{i,j,k,l} |R_{ijk\bar{l}}|^2$$

Since $R_{XYZ\bar{W}}^h = 0$, the above notation is meaningful. The next theorem shows how $|R^h - R|_\star$ measures how much a metric fails to be Kähler. Recall that one possible definition of the Kähler condition is that the torsion identically vanishes.

Theorem. *The following inequalities hold pointwise:*

$$||T||^2 \leq |R^h - R|_\star \text{ and } ||\eta||^2 \leq |R^h - R|_\star$$

Proof. From equation 41 of [18],

$$\frac{1}{2}(R_{X\bar{X}Y\bar{Y}}^h + R_{Y\bar{Y}X\bar{X}}^h) - R_{X\bar{Y}Y\bar{X}} = \sum_k (|T_{XY}^k|^2 + 2\operatorname{Re}(T_{kY}^Y \overline{T_{kX}^X}))$$

We also have that $R_{XY\bar{X}\bar{Y}} = R_{X\bar{X}Y\bar{Y}} - R_{X\bar{Y}Y\bar{X}}$. Therefore, we can rewrite the left-hand side of the above equation:

$$\begin{aligned} & \sum_k (|T_{XY}^k|^2 + 2\operatorname{Re}(T_{kY}^Y \overline{T_{kX}^X})) \\ &= \frac{1}{2}(R_{X\bar{X}Y\bar{Y}}^h + R_{Y\bar{Y}X\bar{X}}^h) - R_{X\bar{X}Y\bar{Y}} - R_{XY\bar{X}\bar{Y}} \\ &= \frac{1}{2}(R_{X\bar{X}Y\bar{Y}}^h - R_{X\bar{X}Y\bar{Y}}) + \frac{1}{2}(R_{Y\bar{Y}X\bar{X}}^h - R_{Y\bar{Y}X\bar{X}}) - R_{XY\bar{X}\bar{Y}} \end{aligned}$$

We want to gain a better understanding of $\sum_{k=1}^n 2\operatorname{Re}(T_{kX}^X \overline{T_{kY}^Y})$. Choose a unitary frame and let $\sum_i T_{ki}^i = \eta_k$.

$$\begin{aligned} \sum_i \sum_j \operatorname{Re}(T_{ki}^i \overline{T_{kj}^j}) &= \operatorname{Re}(\sum_i T_{ki}^i (\sum_j \overline{T_{kj}^j})) \\ &= \operatorname{Re}(\sum_i T_{ki}^i (\bar{\eta}_k)) \\ &= \operatorname{Re}(\bar{\eta}_k \sum_i T_{ki}^i) \\ &= \operatorname{Re}(\bar{\eta}_k \eta_k) \\ &= |\eta_k|^2 \end{aligned}$$

That is to say,

$$\sum_i \sum_j 2\operatorname{Re}(T_{ki}^i \overline{T_{kj}^j}) = 2|\sum_i T_{ki}^i|^2 = 2|\eta_k|^2$$

Thus, we have the following:

$$\begin{aligned}
& \sum_i \sum_j \frac{1}{2} (R_{i\bar{i}j\bar{j}}^h - R_{i\bar{i}j\bar{j}}) + \frac{1}{2} (R_{j\bar{j}i\bar{i}}^h - R_{j\bar{j}i\bar{i}}) - R_{ij\bar{i}\bar{j}} \\
&= \sum_i \sum_j \sum_k (|T_{ij}^k|^2 + 2\operatorname{Re}(T_{ki}^i \overline{T_{kj}^j})) \\
&= \sum_k \sum_i \sum_j (|T_{ij}^k|^2 + 2\operatorname{Re}(T_{ki}^i \overline{T_{kj}^j})) \\
&= \sum_k \left(\left(\sum_i \sum_j |T_{ij}^k|^2 \right) + 2|\eta_k|^2 \right) \\
&= \|T\|^2 + 2\|\eta\|^2
\end{aligned}$$

This then immediately proves that $\|T\|^2 \leq |R^h - R|_\star$. \square

There are many terms in $|R^h - R|_\star$ that are not needed to control the torsion, but we define $|R^h - R|_\star$ as such in view of the next lemma.

Lemma 9. *The following inequality holds where $\nabla^{c''}$ is the derivative with respect to the Chern connection of a $(0,1)$ vector: $|\nabla^{c''} T| \leq |R^h - R|_\star$*

Proof. We use the first equation of Lemma 7 and the following Bianchi identity:

$$\begin{aligned}
R_{i\bar{j}k\bar{l}} - R_{k\bar{j}i\bar{l}} &= R_{i\bar{j}k\bar{l}} + R_{\bar{j}ki\bar{l}} \\
&= -R_{ki\bar{j}\bar{l}} = R_{ik\bar{j}\bar{l}}
\end{aligned}$$

Combining these we find that:

$$e_{\bar{l}}(T_{ik}^j) = (R_{i\bar{j}k\bar{l}}^h - R_{i\bar{j}k\bar{l}}) - (R_{k\bar{j}i\bar{l}}^h - R_{k\bar{j}i\bar{l}}) - R_{ik\bar{j}\bar{l}}$$

\square

We can also bound the derivative with respect to a $(1,0)$ vector.

Lemma 10. *The following inequality holds where $\nabla^{c'}$ is the derivative with respect to the Chern connection of a $(1,0)$ vector:*

$$|\nabla^{c'} T| \leq C(n)|R^h - R|_\star + |R^h - R|_{\star\star}$$

This is an immediate consequence of the equation from Theorem 4 and the following equation from [18]:

$$T_{ij,k}^l = R_{ijk\bar{l}} + T_{rj}^l T_{ik}^r - T_{ri}^l T_{jk}^r$$

Using a straightforward computation, we can relate the derivatives of the torsion tensor with respect to the Levi-Civita connection to the derivatives of the torsion tensor with respect to the Chern connection and a quadratic expression in torsion. Therefore, we also have the following result:

Theorem. *Let T be the torsion tensor and ∇T the derivative of the torsion tensor with respect to the Levi-Civita connection. Then there exists a constant $C'(n)$ that grows at most linearly in n such that the following inequality holds:*

$$\|\nabla T\| \leq C'(n)|R^h - R|_{\star} + |R^h - R|_{\star\star}$$

In future preprints, we will explore these inequalities in much greater depth. One can derive monotonicity results and other structure theorems by leveraging the equations we have used in this paper. However, for the purposes of bounding the principal eigenvalue, what we've done is sufficient.

4. AN ESTIMATE ON THE PRINCIPAL EIGENVALUE OF THE COMPLEX LAPLACIAN

We now combine the two theorems from the previous section with our estimate from the section before.

Theorem. *Let (M^n, h) be a compact Hermitian manifold without boundary with Riemannian curvature satisfying $\text{Ric} \geq -(n-1)k$ for $k \geq 0$ and $\text{diam}(M) = d$. Consider the complex Laplacian \square . Let u satisfy $\square u = \lambda u$ and $K = n^2(k + |R - R^h|_{\star} + |R - R^h|_{\star\star})d^2$. Then there exists an uniform $C > 0$ such that the following estimate holds:*

$$\lambda \geq \frac{1}{4nd^2} \frac{(1 + \sqrt{1 + 4CK})^2 - CK}{\exp(1 + \sqrt{1 + 4CK})}$$

We can simplify this estimate. There exists an uniform $C > 0$ such that:

$$\lambda \geq \frac{1}{4n} \frac{(\frac{2}{d^2} + 3Cn^2(k + |R - R^h|_{\star} + |R - R^h|_{\star\star}))}{\exp(1 + \sqrt{1 + 4CK})}$$

This is still complicated but it only involves the dimension n , the diameter d , a lower bound on the Ricci curvature k , and a measure of how much the metric fails to be Kähler involving only curvature. From the point of view of this estimate, the last term acts in the same way as the Ricci curvature term.

In a future preprint, we will explore monotonicity formulae to show that in some ways the Hermitian curvature dominates the Riemannian curvature. For instance, it is well known that the Hermitian scalar curvature dominates the Riemannian scalar curvature. However,

our monotonicity results are not yet strong enough to bound the spectrum solely using the Riemannian curvature. Nonetheless, this may be possible in light of a theorem due to Yang and Zheng, which states that if the Riemannian curvature is Gray-Kähler-like and the space is compact, then the metric is balanced. Since balanced metrics satisfy $2\Box = \Delta$ on functions, the two operators have the same spectrum. This provides a condition which only depends on the Riemannian curvature and the diameter (more precisely that the diameter is finite), that ensures that spectral geometry of the complex laplacian is the same as that of the Laplace-Beltrame operator. However, we cannot hope to control the entire torsion tensor solely using the Riemannian curvature because there exists Riemannian flat metrics that do not have vanishing torsion (such as the tori in [2]). Also, the example in [18] of a non-compact Gray-Kähler-like surface that is not Kähler shows that Gray-Kähler-like alone does not imply balanced.

Conjecture 11. *Given a compact complex manifold (M^n, h) , there exists C depending only on the dimension, the diameter of M , a lower bound of the Ricci curvature, the Riemannian curvature tensor and the volume of M such that if $\Box u = \lambda u$, then $\lambda \geq C$.*

This conjecture may be more accessible in the conformally balanced or conformally Kähler case, in which case the analysis done on the Yamabe problem may be useful. In the future, we hope to analyze this problem further, leveraging a Harnack-type result using Lemma 7 of [18] in the compact case to study the conjecture. Such an estimate, if it exists, will probably be substantially weaker. More generally, we would like to study how much one can compute about a complex structure on a Hermitian manifold solely from the Riemannian geometry. One goal would be the above conjecture, but an understanding of the moduli space of orthogonal complex structures would be even better.

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